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Existence of equilibria in articulated bearings

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Abstract

The existence of equilibrium solutions for a lubricated system consisting of an articulated body sliding over a flat plate is considered. Though this configuration is very common (it corresponds to the popular tilting-pad thrust bearings), the existence problem has only been addressed in extremely simplified cases, such as planar sliders of infinite width. Our results show the existence of at least one equilibrium for a quite general class of (nonplanar) slider shapes. We also extend previous results concerning planar sliders.

Keywords: Lubrication; Tilting-pad bearing; Equilibrium; Reynolds equation

1. Introduction

Lubricated contacts are thoroughly used in mechanical systems to connect solid bodies that are in relative motion. A fluid, the lubricant, is introduced in the narrow space between the bodies with the purpose of avoiding direct solid-to-solid contact, thus reducing wear of the surfaces and friction losses. When no direct contact takes place, the contact is said to be in the *hydrodynamic* regime and the force transmitted between the bodies results from the shear and pressure forces that develop in the lubricant film.

The lubricated system we consider in this article is perhaps the simplest one. It consists of two nonparallel surfaces in hydrodynamic contact. The bottom surface, assumed planar and horizon-

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tal, moves with a constant horizontal translation velocity. A vertical force $F > 0$ is applied on the upper body. The wedge between the two surfaces is filled with an incompressible fluid. This wedge is assumed to satisfy the thin-film hypotheses, so that the fluid pressure does not depend on the vertical coordinate and obeys the Reynolds equation [2,5,8] which reads, in nondimensional form (and assuming time independence):

$$\nabla \cdot \left[h^3(x)\nabla p\right] = \frac{\partial h}{\partial x_1}, \quad x \in \Omega, \qquad p = 0, \quad x \in \partial\Omega, \tag{1.1}$$

where h is the nondimensional distance between the surfaces and Ω the two-dimensional region in which the hydrodynamic contact takes place. On the boundary of Ω the pressure is equal to the atmospheric pressure which is assumed to be equal to zero. We treat here the case in which h is non-increasing with x_1. This guarantees positivity of the pressure and thus the absence of so-called cavitated regions in which air entrainment or oil degassing would render the model (1.1) invalid [2,6]. An extension of the results presented here to a model that accounts for cavitation has already been established and will be the subject of a forthcoming paper. Nonlinear variants of model (1.1) exist that account for more complex phenomena such as compressibility and rarefaction effects. Such models have been mathematically analyzed by Chipot and Luskin [4], by Chipot [3] and by Buscaglia et al. [1]. Extending the results of this article to nonlinear models has not yet been analyzed and is left for future work.

The upper body, which essentially slides over the bottom planar surface, is sometimes and not surprisingly called a *slider*. The slider-plane configuration we consider is in fact very common in rotating machinery, practically every rotating axis has a *thrust bearing*, which consists of several sliders, to keep it in place resisting axial forces.

Exact solutions for the case in which the upper surface is an inclined plane of angle θ and infinite width have been known for long time, since the problem becomes one-dimensional and is easily integrated [5]. For each given θ the vertical position of the upper surface can be calculated as a function of the applied load, determining its *equilibrium position*. As expected, the two surfaces come closer as the load is increased. The angle that for a given load minimizes the viscous dissipation between the surfaces (or the angle that maximizes the distance between the surfaces, which turns out to be quite close) can be found in engineering texts [2,5,6]. In fact, it is known that this optimal angle is a *strictly decreasing function of the applied load*, implying that fixed-angle sliders only perform well when the applied load is always the same. The performance of a slider designed for some given load quickly deteriorates as the load is varied. This gave rise to the concept of *articulated* sliders, in particular *tilting-pad* thrust bearings.

Articulated sliders have two degrees of freedom: The first one is the vertical displacement a under the effect of the force F and of the pressure load $\int_\Omega p\,dx$. This force is applied by means of an articulated joint parallel to x_2 at a position x_1^0. The second degree of freedom is then the tilt (or pitch) angle θ (see Fig. 1). The slider's vertical position and tilt angle result, of course, of the equilibrium of forces and moments acting on it, and can be found in engineering texts. In particular, an equilibrium condition exists for all applied loads if x_1^0 is located in the downstream half of the slider.

For more general shapes of the upper surface, however, little can be found in the literature concerning articulated sliders if we leave apart the one-dimensional planar case [2,5] and scattered numerical results for particular cases [5]. The purpose of this article is to study the *existence* of equilibria for articulated sliders of arbitrary shape. The normalized distance between the two surfaces is given by (see Appendix A for details)

$$h(x) = h_0(x) + a + \theta x_1,$$

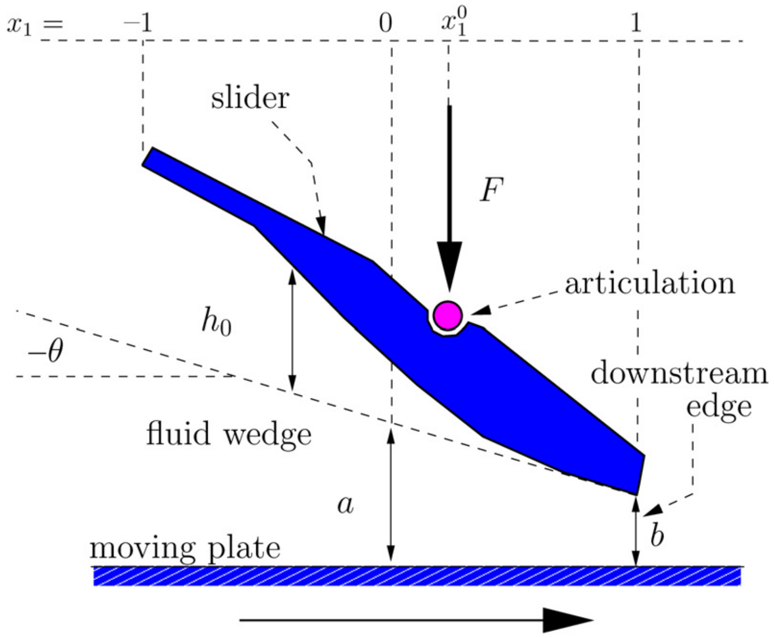

Fig. 1. Scheme of an articulated slider, containing definitions of the different components of the device and some geometrical variables and nondimensional coordinates used in the text.

where $h_0 : \Omega \to \mathbb{R}$ is a given function that accounts for the shape of the slider's surface. The coupled system that results for the *equilibrium solution* consisting of the unknowns $(p, a, \theta) \in H_0^1(\Omega) \times \mathbb{R} \times \mathbb{R}$ reads

$$\nabla \cdot \left[(h_0 + a + \theta x_1)^3 \nabla p\right] = \frac{\partial h_0}{\partial x_1} + \theta \quad \text{in } \Omega, \tag{1.2}$$

$$p = 0 \quad \text{on } \partial\Omega, \tag{1.3}$$

$$\int_\Omega p\, dx = F, \tag{1.4}$$

$$\int_\Omega p\left(x_1 - x_1^0\right) dx = 0. \tag{1.5}$$

Hereafter we will assume that $\Omega =]-1, 1[$ in the 1D case and that $\Omega =]-1, 1[^2$ in the 2D case, with the extensions to more general domains discussed later on. To unify the notation, we denote by x an arbitrary point of Ω, so that $x = x_1$ or $x = (x_1, x_2)$ depending on the case under consideration. Accordingly, dx stands for either dx_1 or $dx_1\, dx_2$.

Recently Ciuperca et al. have analyzed the asymptotic behavior of solutions of (1.2), (1.3) as the minimum of h tends to zero [7]. In this article we assume that h_0 satisfies conditions that guarantee that $\int_\Omega p\, dx$ becomes arbitrarily large as the surfaces approach [7]. These conditions are introduced as hypotheses at the end of this introductory section. Several necessary preliminary results are then introduced in Section 2, some of which are slightly refined, specifically adapted versions of results already proved in [7].

Let us remark that, if θ is fixed, p clearly tends to zero as $a \to +\infty$. This, together with the continuity of p with respect to a and the hypotheses mentioned in the previous paragraph, implies the existence of an equilibrium value of a for which (1.4) is satisfied. As soon as θ becomes a degree of freedom the situation is much less obvious even in the 1D case, which is dealt with in Section 3. There it is proved that, for any given $F > 0$, there exists a position $\bar{x}_1$ such that if x_1^0 belongs to $]\bar{x}_1, 1[$ then there exists an equilibrium solution of (1.2)–(1.5). This same result is extended to the 2D case in Section 4. We conclude the article with Section 5 revisiting the case in which the slider's surface in planar, to show that in this case $\bar{x}_1$ can easily be determined and *does not depend on F*.

As mentioned, we conclude this section establishing the hypotheses assumed for the shape function h_0:

$$h_0 \in W^{1,\infty}(\Omega), \quad \frac{\partial h_0}{\partial x_1}(x) \leqslant 0 \quad \text{a.e. in } \Omega, \quad h_0(x) \geqslant 0, \tag{H$_1$}$$

$$\exists \alpha > 1,\ h_1 \in W^{1,\infty}(\Omega) \quad \text{such that} \quad h_0(x) = (1 - x_1)^\alpha h_1(x). \tag{H$_2$}$$

To ensure positivity of p, we will look for a and θ such that

$$h_0(x) + a + \theta x_1 > 0 \quad \forall x \in \Omega, \tag{1.6}$$

$$\frac{\partial}{\partial x_1}(h_0 + a + \theta x_1) \leqslant 0 \quad p.p.\ x \in \Omega. \tag{1.7}$$

In fact, (1.7) is equivalent to

$$\theta \leqslant \inf \mathrm{ess}_{x \in \Omega} \left\{ -\frac{\partial h_0}{\partial x_1}(x) \right\}.$$

From (H$_2$) we have $\inf \mathrm{ess}_{x\in\Omega}\{-\frac{\partial h_0}{\partial x_1}(x)\} = 0$. Then (1.6) is equivalent to $a + \theta > 0$. We will thus look for a and θ satisfying

$$\begin{cases} \theta < 0, \\ a + \theta > 0. \end{cases} \tag{1.8}$$

2. Preliminaries

In this section, we introduce the following auxiliary problem with unknown $P : \Omega \to \mathbb{R}$.

$$\begin{cases} \nabla \cdot \left[(H_0 + A)^3 \nabla P \right] = \frac{\partial H_0}{\partial x_1}, & x \in \Omega, \\ P = 0, & x \in \partial\Omega \end{cases} \tag{2.1}$$

with $A > 0$ and H_0 satisfying

$$\begin{cases} H_0 \in W^{1,\infty}(\Omega), \quad \min_{x \in \overline{\Omega}} H_0(x) = 0, \\ \frac{\partial H_0}{\partial x_1}(x) \leqslant 0 \quad \text{a.e. in } \Omega, \qquad \mathrm{meas}\left\{ x \in \Omega\colon \frac{\partial H_0}{\partial x_1}(x) < 0 \right\} > 0. \end{cases} \tag{2.2}$$

Let us introduce the following application

$$G\colon \quad]0, +\infty[\to \mathbb{R}, \quad A \mapsto G(A) = \int_\Omega P\, dx$$

with $P(x) = P_A(x)$ the (unique) solution of (2.1). Hypotheses (2.2) imply $P \geqslant 0$, thus $G(A) \geqslant 0$. We also have, from (2.2), $G(A) > 0$ for all $A > 0$. The results below characterize the behavior of the function $G(A)$.

Proposition 2.1. *G is of class C^∞.*

Proof. We apply the implicit function theorem to $L : H_0^1(\Omega) \times]0, +\infty[\to H^{-1}(\Omega)$

$$L(Q, B) = -\nabla \cdot \left[(H_0 + B)^3 \nabla Q\right] + \frac{\partial H_0}{\partial x_1}.$$

It is clear that L is of class C^∞ and that

$$\frac{\partial L}{\partial Q}(P_A, A) \cdot Z = -\nabla \cdot \left[(H_0 + A)^3 \nabla Z\right],$$

where we denote by P_A the unique solution P of (2.1). This implies that $\frac{\partial L}{\partial P}(P_A, A)$ is an isomorphism from $H_0^1(\Omega)$ onto $H^{-1}(\Omega)$. The function

$$A \in]0, +\infty[\to P_A \in H_0^1(\Omega)$$

is thus of class C^∞. □

Proposition 2.2. *We have*

$$\lim_{A \to +\infty} G(A) = 0 \quad \textit{with} \quad G(A) \leqslant \frac{c}{A^3} \|H_0\|_{L^2(\Omega)},$$

where $c > 0$ is a constant independent of A and of H_0.

Proof. Taking P as a test function in the variational formulation of (2.1) we have

$$\int_\Omega (H_0 + A)^3 |\nabla P|^2 \, dx = \int_\Omega H_0 \frac{\partial P}{\partial x_1} \, dx$$

which gives, since $H_0 \geqslant 0$,

$$\|\nabla P\|_{L^2(\Omega)} \leqslant \frac{1}{A^3} \|H_0\|_{L^2(\Omega)}.$$

The result then follows from Poincaré inequality. □

Let us introduce, for $\eta \in]0, 2]$

$$\Omega_\eta =]1 - \eta, 1[\times]-1, 1[\quad \text{for } n = 2 \quad \text{and} \quad \Omega_\eta =]1 - \eta, 1[\quad \text{for } n = 1.$$

Proposition 2.3. *Assume, in addition to hypotheses* (2.2), *that there exist* $0 < M_0 < M_1$, $\alpha \geqslant 1$ *and* $\delta_0 \in]0, 1[$ *such that*

$$M_0(1 - x_1)^{\alpha - 1} \leqslant -\frac{\partial H_0}{\partial x_1}(x) \leqslant M_1(1 - x_1)^{\alpha - 1} \quad \textit{a.e. } x \in \Omega_{2\delta_0}. \tag{2.3}$$

Then there exists $c = c(M_0, M_1, \delta_0, \alpha)$ such that the solution P of (2.1) *satisfies*

$$\int_{\Omega_B} P(x) \, dx \geqslant c\left(\log\left(1 + M_1 \frac{B}{A}\right) - 2\right) \quad \forall B \in]0, \delta_0] \; \forall A > 0, \quad \textit{if } \alpha = 1 \quad \textit{and}$$

$$\int_{\Omega_B} P(x) \, dx \geqslant c A^{\frac{2}{\alpha} - 2} \quad \forall B \in]0, \delta_0] \; \forall A < B^\alpha, \quad \textit{if } \alpha > 1.$$

Proof. Let us begin the proof with the two-dimensional case ($n = 2$). We remark by writing

$$H_0(x) = \int_{x_1}^{1} \left[-\frac{\partial H_0}{\partial x_1}(s, x_2)\, ds \right]$$

and integrating (2.3) that

$$\frac{M_0}{\alpha}(1 - x_1)^\alpha \leqslant H_0(x) \leqslant \frac{M_1}{\alpha}(1 - x_1)^\alpha \quad \text{a.e. in } \Omega_{2\delta_0}.$$

Let $\phi \in C^2[1 - 2\delta_0, 1]$ with $\phi \geqslant 0$, $\phi(1 - 2\delta_0) = 0$ and $\phi(x) = 1, \forall x \in [1 - \delta_0, 1]$, $q_2(x_2) = 1 - x_2^2$ and

$$q_1(x_1) = \frac{(1 - x_1)^{\alpha+1}}{(M_1(1 - x_1)^\alpha + A)^3}.$$

From the asymptotic analysis results of Ciuperca et al. [7, Lemma 3.6] there exists $c_1 = c_1(M_0, M_1, \delta_0, \alpha) > 0$ such that

$$P(x) \geqslant c_1 \phi(x_1) q_1(x_1) q_2(x_2) \quad \forall x \in \Omega_{2\delta_0}.$$

This implies

$$\int_{\Omega_B} P(x)\, dx \geqslant \frac{4}{3} c_1 \int_{1-B}^{1} q_1(x_1)\, dx_1.$$

For $\alpha > 1$ we have, for $A < B^\alpha$,

$$\int_{1-B}^{1} q_1(x_1)\, dx_1 \geqslant \frac{4}{3} c_1 \frac{1}{(M + 1)^3 A^3} \int_{1-A^{1/\alpha}}^{1} (1 - x_1)^{\alpha+1}\, dx_1$$

which gives the result. For $\alpha = 1$ let us denote $f_1(x_1) = M_1(1 - x_1) + A$. Then,

$$q_1(x_1) = \frac{1}{M_1^2 f_1^3}\left(f_1^2 - 2Af_1 + A^2\right) \geqslant \frac{1}{M_1^2}\left(\frac{1}{f_1} - \frac{2A}{f_1^2}\right)$$

which upon integration on $[1 - B, 1]$ gives the result. The proof in the one-dimensional case ($n = 1$) is similar, it suffices to take $q_2 = 1$. $\square$

Corollary 2.4. *Under the hypotheses of Proposition* 2.3 *we have*

$$\lim_{A \to 0,\ A > 0} G(A) = +\infty.$$

Proof. Since $P \geqslant 0$ we have, using Proposition 2.3, that

$$\int_{\Omega} P(x)\, dx \geqslant \int_{\Omega_{\delta_0}} P(x)\, dx \geqslant c\left(\log\left(1 + M_1 \frac{\delta_0}{A}\right) - 2\right). \qquad \square$$

Proposition 2.5. *The function G satisfies $\frac{dG}{dA}(A) < 0$, $\forall A > 0$, in the following two situations*:

(i) *In one-dimensional case* ($n = 1$).
(ii) *In the two-dimensional case when H_0 is given by*

$$H_0(x) = C(1 - x_1), \quad \text{with } C > 0 \quad (\text{planar case}).$$

Proof. We have $G'(A) = \int_\Omega Q\,dx$, where $Q = \frac{dP}{dA} \in H_0^1(\Omega)$ is the solution of the following problem:

$$\begin{cases} \nabla \cdot [(H_0 + A)^3 \nabla Q] = -3\nabla \cdot [(H_0 + A)^2 \nabla P] & \text{in } \Omega, \\ Q = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.4}$$

(i) *In the one-dimensional case*. We will show that

$$\big((H_0 + A)^2 P'\big)' \leqslant 0 \quad \text{and} \quad \text{meas}\big\{x \in \Omega\colon \big((H_0 + A)^2 P'\big)' < 0\big\} > 0 \tag{2.5}$$

which implies, from the maximum principle, that $Q \leqslant 0$ and gives us the result. Since we are in the one-dimensional case we can integrate (2.1) explicitly. This gives

$$(H_0 + A)^3 P' = H_0 + A - C(A) \tag{2.6}$$

with

$$C(A) = \frac{\int_{-1}^1 (H_0 + A)^{-2}\,dx_1}{\int_{-1}^1 (H_0 + A)^{-3}\,dx_1} > 0.$$

Dividing (2.6) by $H_0 + A$ and differentiating with respect to x_1 we obtain

$$\big((H_0 + A)^2 P'\big)' = C(A)\frac{H_0'}{(H_0 + A)^2}.$$

Due to (2.2), this last equation implies (2.5). This ends the proof in the one-dimensional case.

(ii) *In the two-dimensional planar case.* Taking Q as test function in (2.1) and using $\frac{\partial H_0}{\partial x_1} = -C$ we have

$$C\int_\Omega Q = \int_\Omega (H_0 + A)^3 \nabla P \cdot \nabla Q\,dx.$$

Now taking P as test function in (2.4) we obtain

$$\int_\Omega (H_0 + A)^3 \nabla P \cdot \nabla Q\,dx = -3\int_\Omega (H_0 + A)^2 |\nabla P|^2\,dx < 0 \quad (\text{since } P \not\equiv 0)$$

which gives the result. □

We conclude this section with a technical lemma. It gives a comparison result between one-dimensional and two-dimensional solutions of some elliptic problems.

Lemma 2.6. *Let $\Omega =]-1,1[\times]-1,1[$. Let $\omega_1 \in L^\infty(]-1,1[)$, $\omega_2 \in W^{1,\infty}(]-1,1[)$, $\xi_1 \in W^{1,\infty}(\Omega)$ and $\xi_2 \in L^\infty(\Omega)$ be such that $\operatorname{inf\,ess}_{x_1 \in]-1,1[}\,\omega_1(x_1) > 0$, $\xi_1(x) \geqslant 0$, a.e. in Ω, $\omega_2(x_1) \geqslant 0$ and $\omega_2'(x_1) \leqslant 0$ for a.e. $x_1 \in]-1,1[$. Let u be the solution of the following problem*

$$\begin{cases} \nabla \cdot [\omega_1(1 + \xi_1)\nabla u] = \omega_2' + \xi_2, & x \in \Omega, \\ u \in H_0^1(\Omega). \end{cases} \tag{2.7}$$

Assume that ξ_1 *and* ξ_2 *are small in the following sense*: *There exists* $\gamma > 0$ *such that*

$$\gamma\omega_2(x_1)\frac{\partial\xi_1}{\partial x_1}(x) - \xi_2(x) \leqslant -(\gamma-1)\omega_2'(x_1) \quad \text{a.e. in } \Omega. \tag{2.8}$$

Then

$$u(x) \leqslant \gamma v(x_1) \quad \text{a.e. in } \Omega,$$

where

$$v(x_1) = \int_{-1}^{x_1} \frac{\omega_2(s)}{\omega_1(s)}\,ds \quad \forall x_1 \in]-1,1[.$$

Proof. We apply the maximum principle. Since $v \geqslant 0$ we have

$$u(x) \leqslant \gamma v(x_1) \quad \text{for a.e. } x \in \partial\Omega.$$

It suffices then to prove that

$$-\gamma\frac{\partial}{\partial x_1}\left[\omega_1(1+\xi_1)\frac{dv}{dx_1}\right] \geqslant -\omega_2' - \xi_2 \quad \text{for a.e. } x \in \Omega.$$

Since v satisfies $\omega_1 v' = \omega_2$, the previous inequality is equivalent to

$$-(\gamma-1)\omega_2'(x_1) - \gamma\xi_1\omega_2' - \gamma\omega_2\frac{\partial\xi_1}{\partial x_1} \geqslant -\xi_2$$

which is true from the hypothesis, since $\gamma\,\xi_1\omega_2' \leqslant 0$. □

3. The one-dimensional case ($n = 1$)

We look for $(p(x), a, \theta)$ satisfying

$$\big((h_0 + a + \theta x)^3 p'\big)' = (h_0 + a + \theta x)' \quad \text{in } \Omega =]-1,1[, \tag{3.1}$$

$$p(-1) = p(1) = 0, \tag{3.2}$$

$$\int_\Omega p\,dx = F, \tag{3.3}$$

$$\int_\Omega p\big(x_1 - x_1^0\big)\,dx = 0, \tag{3.4}$$

$$a + \theta > 0, \tag{3.5}$$

$$\theta < 0 \tag{3.6}$$

with

$$h_0(x) + a + \theta x = a + \theta + (1-x)\big(r(x) - \theta\big),$$

where $r(x) = \frac{h_0(x)}{1-x}$. To apply the results of Section 2 we make the change of variables

$$(b, \theta) = (a + \theta, \theta).$$

Problem (3.1)–(3.6) thus becomes

$$\bigl(\bigl(b+(1-x)\bigl(r(x)-\theta\bigr)\bigr)^3 p'\bigr)' = \bigl((1-x)\bigl(r(x)-\theta\bigr)\bigr)' \quad \text{in } \Omega, \tag{3.7}$$

$$p(-1) = p(1) = 0, \tag{3.8}$$

$$\int_\Omega p\,dx = F, \tag{3.9}$$

$$\int_\Omega p\bigl(x_1 - x_1^0\bigr)\,dx = 0, \tag{3.10}$$

$$b > 0, \tag{3.11}$$

$$\theta < 0. \tag{3.12}$$

Solving (3.7)–(3.12) consists of finding $b > 0$ and $\theta < 0$ satisfying

$$\begin{cases} g_1(b,\theta) = 0, \\ g_2(b,\theta) = 0, \end{cases} \tag{3.13}$$

where $g_1, g_2 :]0,+\infty[\times]-\infty, 0[\to \mathbb{R}$ are defined as

$$g_1(b,\theta) = \int_\Omega p\,dx - F, \qquad g_2(b,\theta) = \int_\Omega p\bigl(x_1 - x_1^0\bigr)\,dx$$

with $p = p(b,\theta)$ the unique solution of (3.7), (3.8) for (b,θ) given.

Proposition 3.1. *The functions g_1 and g_2 are of class C^∞ in $]0,+\infty[\times]-\infty, 0[$.*

Proof. It is a direct consequence of the implicit functions theorem, as in Proposition 2.1. □

Proposition 3.2. *For all $\theta < 0$ there exists a unique $b > 0$ such that $g_1(b,\theta) = 0$.*

Proof. We need to prove the existence and uniqueness of a solution (p, b) of (3.7)–(3.9). Applying Propositions 2.1 and 2.2 with $H_0(x) = (1-x)(r(x)-\theta)$ and $A = b$ we have that the application $b \in]0,+\infty[\to g_1(b,\theta)$ is continuous and that $\lim_{b\to+\infty} g_1(b,\theta) = -F < 0$. We also have

$$-H_0'(x) = -h_0'(x) - \theta$$

and since $\theta < 0$ and $h_0 \in C^1(\overline{\Omega})$ the hypotheses of Proposition 2.3 and of Corollary 2.4 hold. As a consequence, $\lim_{b\to 0} g_1(b,\theta) = +\infty$. The intermediate value theorem then gives us the existence and the monotonicity (Proposition 2.5) with $n = 1$ implies the uniqueness. □

Let $b(\theta)$ be the solution found in Proposition 3.2. We define the application

$$S : \theta \in]-\infty, 0[\to S(\theta) = g_2\bigl(b(\theta), \theta\bigr). \tag{3.14}$$

Finding a solution of (3.13) amounts thus to finding θ such that

$$S(\theta) = 0.$$

Proposition 3.3. *The application S is of class C^∞.*

Proof. We apply Proposition 2.5(i) with $H_0(x) = (1-x)(r(x)-\theta)$ and $A = b$ to get $\frac{\partial g_1}{\partial b}(b(\theta),\theta) < 0$, $\forall \theta < 0$. Since $g_1 \in C^\infty$ the implicit function theorem implies that the application $\theta \in]-\infty, 0[\to b(\theta)$ is of class C^∞. This, together with $g_2 \in C^\infty$ implies the result. □

Lemma 3.4. *For all $\theta < 0$ and $b > 0$ the solution p of* (3.7), (3.8) *satisfies*

$$p(x) \leqslant \int_{-1}^{x} \frac{ds}{[b + (1-s)(r(s)-\theta)]^2}.$$

Proof. We apply the maximum principle. Denoting

$$q(x) = \int_{-1}^{x} \frac{ds}{[b + (1-s)(r(s)-\theta)]^2}$$

it is clear that q satisfies the same equation as p and that $q(x) > p(x) = 0$ for $x \in \partial\Omega = \{-1, 1\}$. □

We are now in a position to state our existence result.

Theorem 3.5. *For any $F > 0$ there exists $\bar{x} \in]-1, 1[$ such that $\forall x_1^0 \in]\bar{x}, 1[$ there exists at least one solution $(p(x), b, \theta)$ of system* (3.7)–(3.12).

Proof. We need to prove that the function S vanishes for some θ. The proof is carried out in two steps.

Step 1. We first show that

$$\lim_{\theta \to -\infty} S(\theta) > 0. \tag{3.15}$$

Using definition (3.14) of S and the identity $\int_{-1}^{1} p\,dx = F$ we rewrite $S(\theta)$ as

$$S(\theta) = \left(1 - x_1^0\right)F - \int_{-1}^{1} (1-x)p\big(b(\theta),\theta\big)\,dx. \tag{3.16}$$

From Lemma 3.4 and since $r > 0$ we have

$$p(x) \leqslant \int_{-1}^{x} \frac{ds}{[b - \theta(1-s)]^2}$$

and an elementary calculation yields

$$p(x) \leqslant \frac{1}{-\theta}\left[\frac{1}{b - \theta(1-x)} - \frac{1}{b - 2\theta}\right] \leqslant \frac{1}{(1-x)(\beta_1 - \theta)^2}.$$

We thus have

$$0 \leqslant \int_{-1}^{1} (1-x)p(x)\,dx \leqslant \frac{2}{\theta^2}$$

implying that

$$\lim_{\theta\to-\infty}\int_{-1}^{1}(1-x)p(x)\,dx=0.$$

Relation (3.16) then gives (3.15) because $x_1^0<1$ and $F>0$.

Step 2. It remains to show that there exists $\theta_0<0$ such that $S(\theta_0)<0$. Since $-p(x)<xp(x)<p(x)$ for all x such that $p(x)>0$ we obtain by integration

$$-\int_\Omega p\,dx<\int_\Omega xp\,dx<\int_\Omega p\,dx.$$

Let ξ^θ be defined by

$$\xi^\theta=\frac{\int_\Omega xp\,dx}{\int_\Omega p\,dx}\in\,]-1,1[.$$

We then have

$$S(\theta)=g_2\bigl(b(\theta),\theta\bigr)=\bigl(\xi^\theta-x_1^0\bigr)\int_\Omega p(x)\,dx=\bigl(\xi^\theta-x_1^0\bigr)F.$$

Putting $\bar{x}=\inf_{\theta<0}\xi^\theta\in[-1,1[$ the result is immediate. $\square$

4. The two-dimensional case ($n=2$)

We consider now problem (1.2)–(1.5) subject to conditions (1.6), (1.7). With the same change of variables as before, $(b,\theta)=(a+\theta,\theta)$, it becomes

$$\nabla\cdot\Bigl(\bigl(b+(1-x_1)\bigl(r_2(x)-\theta\bigr)\bigr)^3\nabla p\Bigr)=\frac{\partial}{\partial x_1}\bigl((1-x_1)\bigl(r_2(x)-\theta\bigr)\bigr)\quad\text{in }\Omega,\tag{4.1}$$

$$p=0\quad\text{on }\partial\Omega,\tag{4.2}$$

$$\int_\Omega p\,dx=F,\tag{4.3}$$

$$\int_\Omega p\bigl(x_1-x_1^0\bigr)\,dx=0,\tag{4.4}$$

$$b>0,\tag{4.5}$$

$$\theta<0\tag{4.6}$$

with $r_2(x)=\frac{h_0(x)}{1-x_1}$. We denote

$$\begin{cases}M_2=\sup_{x\in\Omega}r_2(x)>0,\\ M_3=\sup_{x\in\Omega}\frac{h_0(x)}{(1-x_1)^\alpha}.\end{cases}\tag{4.7}$$

Solving system (4.1)–(4.6) thus amounts to solving

$$g_1(b,\theta) \equiv \int_\Omega p\,dx - F = 0, \tag{4.8}$$

$$g_2(b,\theta) \equiv \int_\Omega p\big(x_1 - x_1^0\big)\,dx = 0 \tag{4.9}$$

with $p = p(b,\theta)$ the solution of (4.1), (4.2). It is shown as before that g_1 and g_2 are of class C^∞ and we have, as in Proposition 3.2, the following existence result for b with θ given.

Proposition 4.1. *For all $\theta < 0$ there exists $b > 0$ such that*

$$g_1(b,\theta) = 0.$$

The uniqueness of such a b is however not obvious since the monotonicity of g_1 is not guaranteed in two dimensions (at least, we know of no such result and could not prove it). The monotonicity does hold, however, if $r_2(x) - \theta$ is a constant (Proposition 2.5(ii)). The proof below is thus based on establishing suitable bounds so as to be able to neglect $r_2(x)$ for $-\theta$ large enough. Dividing (4.1) by $-\theta^3$ and denoting

$$\hat b = -\frac{b}{\theta}, \qquad \hat p = \theta^2 p, \tag{4.10}$$

$$f_{\hat b,\theta}(x) = \hat b + 1 - x_1 - \frac{h_0(x)}{\theta} \tag{4.11}$$

problem (4.1)–(4.3) with θ given is equivalent to: *Find $(\hat p, \hat b)$ satisfying*

$$\nabla\cdot\big[f_{\hat b,\theta}(x)^3\nabla\hat p\big] = \frac{\partial}{\partial x_1}\big[f_{\hat b,\theta}(x)\big] \quad \text{in } \Omega, \tag{4.12}$$

$$\hat p = 0 \quad \text{on } \partial\Omega, \tag{4.13}$$

$$\int_\Omega \hat p\,dx = \theta^2 F. \tag{4.14}$$

Lemma 4.2. *For all $\hat b > 0$ and $\theta \leqslant -25\|\frac{\partial h_0}{\partial x_1}\|_\infty$ the solution $\hat p$ of* (4.12), (4.13) *satisfies*

$$\hat p(x) \leqslant \frac{2}{\hat b + 1 - x_1} \quad \forall x \in \Omega.$$

Proof. Problem (4.12), (4.13) is a particular case of (2.7) (Lemma 2.6) with $\omega_1 = (1 - x_1 + \hat b)^3$, $\omega_2 = 1 - x_1 + \hat b$, $\xi_1 = (1 - \frac{h_0}{\theta\omega_2})^3 - 1$ and $\xi_2 = -\frac{1}{\theta}\frac{\partial h_0}{\partial x_1}$. The basic hypotheses of Lemma 2.6 are, in fact, verified: $\inf_{x_1\in]-1,1[}\omega_1(x_1) = \hat b^3 > 0$, $\omega_2 > 0$, $\omega_2' = -1$ and $\xi_1 \geqslant 0$ (because $\theta < 0$). It remains to verify the smallness hypothesis (2.8). Taking $\gamma = 2$ we will show that

$$2\omega_2\frac{\partial\xi_1}{\partial x_1} - \xi_2 \leqslant 1.$$

In fact,

$$2\omega_2\frac{\partial\xi_1}{\partial x_1} = \frac{6}{|\theta|}\frac{\partial h_0}{\partial x_1}\left(1 - \frac{h_0}{\theta\omega_2}\right)^2 + \frac{6}{|\theta|}\frac{h_0}{\omega_2}\left(1 - \frac{h_0}{\theta\omega_2}\right)^2 \leqslant \frac{6}{|\theta|}\frac{h_0}{1-x_1}\left(1 + \frac{1}{|\theta|}\frac{h_0}{1-x_1}\right)^2.$$

Since from (H_1) we have $\frac{h_0}{1-x_1} \leqslant \|\frac{\partial h_0}{\partial x_1}\|_\infty$ it follows that (2.8) indeed holds. We thus apply Lemma 2.6 to get

$$\hat{p}(x) \leqslant 2\int_{-1}^{x_1} \frac{1}{(1-s+\hat{b})^2}\,ds \leqslant \frac{2}{1-x_1+\hat{b}}. \qquad \square$$

We now need the following lemma.

Lemma 4.3. *There exist $b_0 \in]0,1[$ and $c_0 > 0$ which depend only on h_0 such that*

$$\int_\Omega f^2_{\hat{b},\theta}(x)|\nabla\hat{p}|^2\,dx \geqslant \frac{c_0}{\hat{b}} \quad \forall\theta < \theta_0, \quad \forall\hat{b} \in]0,b_0[,$$

where

$$\theta_0 = \min\left\{-\max_{x\in\overline{\Omega}} r_2(x), -25\left\|\frac{\partial h_0}{\partial x_1}\right\|_\infty\right\}.$$

Proof. Since $\theta < -\max_{x\in\overline{\Omega}} r_2(x)$ we have that

$$\frac{h_0(x)}{|\theta|} \leqslant 1 - x_1 \quad \forall x \in \Omega$$

and thus

$$f_{\hat{b},\theta}(x) \leqslant \hat{b} + 2(1-x_1) \quad \forall x \in \Omega. \tag{4.15}$$

It is then easy to see that, for any $\mu > 0$ such that $\mu\hat{b} \leqslant 1$,

$$\int_\Omega f^2_{\hat{b},\theta}(x)|\nabla\hat{p}|^2\,dx \geqslant \frac{1}{(1+4\mu)\hat{b}} \int_{\Omega_{2\mu\hat{b}}} f^3_{\hat{b},\theta}(x)|\nabla\hat{p}|^2\,dx, \tag{4.16}$$

where we recall the notation

$$\Omega_\eta =]1-\eta,1[\times]-1,1[, \quad \eta \in]0,2].$$

It only remains to show that there exists $\mu > 0$, independent of $\hat{b}$, such that $\int_{\Omega_{2\mu\hat{b}}} f^3_{\hat{b},\theta}(x)|\nabla\hat{p}|^2\,dx$ is bounded from below by a strictly positive constant. For this purpose we consider a truncation function $\psi \in C^2(\mathbb{R})$, $0 \leqslant \psi \leqslant 1$, such that

$$\psi(x_1) = \begin{cases} 0 & \text{if } x_1 \leqslant 0, \\ 1 & \text{if } x_1 \geqslant 1 \end{cases}$$

and we define

$$\psi_{\hat{b}}(x_1) = \psi\left(\frac{x_1 - 1 + 2\mu\hat{b}}{\mu\hat{b}}\right)$$

which satisfies

$$\psi_{\hat{b}}(x_1) = 0, \quad \text{if } x_1 \leqslant 1 - 2\mu\hat{b}, \quad \text{and} \quad \psi_{\hat{b}}(x_1) = 1, \quad \text{if } x_1 \geqslant 1 - \mu\hat{b}.$$

Taking now $\phi = \psi_{\hat{b}}\hat{p}$ as test function in the variational form of (4.12), (4.13), and since

$$-\frac{\partial}{\partial x_1}[f_{\hat{b},\theta}] = 1 + \frac{1}{\theta}\frac{\partial h_0}{\partial x_1}(x)$$

we get

$$\int_\Omega f_{\hat b,\theta}^3 |\nabla \hat p|^2 \psi_{\hat b}\,dx + \frac{1}{2}\int_\Omega f_{\hat b,\theta}^3 \nabla\psi_{\hat b}\cdot\nabla\big(\hat p^2\big)\,dx = \int_\Omega \left(1+\frac{1}{\theta}\frac{\partial h_0}{\partial x_1}\right)\psi_{\hat b}\hat p\,dx$$

which can be expressed as

$$\int_\Omega f_{\hat b,\theta}^3(x)|\nabla\hat p|^2\psi_{\hat b}\,dx = E_1 + E_2 \tag{4.17}$$

with

$$E_1 = \frac{1}{2}\int_\Omega \nabla\cdot\big(f_{\hat b,\theta}^3\nabla\psi_{\hat b}\big)\hat p^2\,dx = \frac{1}{2}\int_{-1}^{1}\int_{1-2\mu\hat b}^{1-\mu\hat b}\frac{\partial}{\partial x_1}\left(f_{\hat b,\theta}^3\frac{d\psi_{\hat b}}{dx_1}\right)\hat p^2\,dx, \tag{4.18}$$

$$E_2 = \int_\Omega\left(1+\frac{1}{\theta}\frac{\partial h_0}{\partial x_1}\right)\psi_{\hat b}\hat p\,dx. \tag{4.19}$$

From the following immediate inequalities

$$f_{\hat b,\theta}(x)\leqslant \hat b(1+4\mu)\quad \forall x_1\geqslant 1-2\mu\hat b,\qquad \left|\frac{\partial}{\partial x_1}f_{\hat b,\theta}\right|\leqslant 1+\frac{1}{|\theta_0|}\left\|\frac{\partial h_0}{\partial x_1}\right\|_\infty,$$
$$\big|\psi_{\hat b}'(x_1)\big|\leqslant\frac{1}{\mu\hat b}\sup_{x_1\in\mathbb{R}}\big|\psi'(x_1)\big|,\qquad \big|\psi_{\hat b}''(x_1)\big|\leqslant\frac{1}{\mu^2\hat b^2}\sup_{x_1\in\mathbb{R}}\big|\psi''(x_1)\big|$$

we get

$$|E_1|\leqslant c_1\hat b\left[\frac{(1+4\mu)^3}{\mu^2}+\frac{(1+4\mu)^2}{\mu}\right]\int_{-1}^{1}\int_{1-2\mu\hat b}^{1-\mu\hat b}\hat p^2\,dx\quad\text{with}$$

$$c_1=\frac{1}{2}\max\left\{3\left(1+\frac{1}{|\theta_0|}\left\|\frac{\partial h_0}{\partial x_1}\right\|_\infty\right)\|\psi'\|_\infty,\|\psi''\|_\infty\right\}.$$

Now, from Lemma 4.2 and since $\theta < -25\|\frac{\partial h_0}{\partial x_1}\|_\infty$, we know that

$$\int_{-1}^{1}\int_{1-2\mu\hat b}^{1-\mu\hat b}\hat p^2\,dx\leqslant\frac{8}{\hat b(1+\mu)}$$

which implies

$$|E_1|\leqslant 8c_1\frac{1+4\mu}{1+\mu}\left[\frac{(1+4\mu)^2}{\mu^2}+\frac{1+4\mu}{\mu}\right]. \tag{4.20}$$

Turning now to E_2, since $\frac{1}{\theta}\frac{\partial h_0}{\partial x_1}(x)\geqslant 0$, $\hat p\geqslant 0$, $\psi_{\hat b}\geqslant 0$ and $\psi_{\hat b}=1$ on $[1-\mu\hat b,1]$ we have that

$$E_2\geqslant\int_{-1}^{1}\int_{1-\mu\hat b}^{1}\hat p\,dx=\int_{\Omega_{\mu\hat b}}\hat p\,dx. \tag{4.21}$$

We now apply Proposition 2.3 with $H_0(x) = 1 - x_1 - \frac{h_0(x)}{\theta}$, $\alpha = 1$, $\delta_0 = 1$ and $A = \hat{b}$. H_0 verifies the hypotheses of Proposition 2.3 because

$$-\frac{\partial H_0}{\partial x_1}(x) = 1 + \frac{1}{\theta}\frac{\partial h_0}{\partial x_1}(x)$$

which gives

$$M_0 = 1 \leqslant -\frac{\partial H_0}{\partial x_1}(x) \leqslant \frac{1}{|\theta_0|}\left\|\frac{\partial h_0}{\partial x_1}\right\|_\infty + 1 = M_1 \quad \forall x \in \Omega.$$

From Proposition 2.3 with $B = \mu\hat{b}$ and from (4.21) we thus get

$$E_2 \geqslant c_2\big[\log(1 + M_1\mu) - 2\big] \tag{4.22}$$

with c_2 independent of $\hat{b}$, θ and μ. We now choose $\mu > 0$ large enough to satisfy

$$c_2\big(\log(1 + M_1\mu) - 2\big) - 8c_1\frac{1+4\mu}{1+\mu}\left[\frac{(1+4\mu)^2}{\mu^2} + \frac{1+4\mu}{\mu}\right] \geqslant 1,$$

for example, we could take

$$\mu = \big(\exp\big((1 + 640c_1 + 2c_2)/c_2\big) - 1\big)/M_1.$$

Notice that this choice of μ is independent of $\hat{b}$ and θ. We take $b_0 > 0$ such that $b_0\mu < 1$ and we deduce, from (4.17)–(4.21) that

$$\int_\Omega f^3_{\hat{b},\theta}(x)|\nabla\hat{p}|^2\psi_{\hat{b}}\,dx \geqslant 1 \quad \forall\hat{b} < b_0.$$

Since $\psi_{\hat{b}} \leqslant 1$ for $x_1 \in [1 - 2\mu\hat{b}, 1]$ and $\psi_{\hat{b}} = 0$ for $x_1 \in [-1, 1 - 2\mu\hat{b}]$,

$$\int_{\Omega_{2\mu\hat{b}}} f^3_{\hat{b},\theta}(x)|\nabla\hat{p}|^2\,dx \geqslant 1$$

which, together with (4.16), proves the claimed result with $c_0 = \frac{1}{1+4\mu}$. $\square$

We can now state the following uniqueness result.

Proposition 4.4. *There exists $\theta_1 < 0$ such that for any $\theta < \theta_1$ there exists a unique b (denoted by $b(\theta)$) satisfying $g_1(b,\theta) = 0$, with g_1 as in Proposition* 4.1. *Moreover, $\frac{\partial g_1}{\partial b}(b(\theta),\theta) < 0$.*

Proof. We organize the proof in two steps.

Step 1. We first show that the function $\hat{b} \in [0, b_0] \to \int_\Omega \hat{p}(\hat{b},\theta)\,dx$ is strictly decreasing for all $\theta < -3^{\alpha-1}\frac{M_3}{\sqrt{c_0(\alpha-1)}}$, with b_0 and c_0 as in Lemma 4.3.

Let us denote $\hat{q} = \frac{\partial\hat{p}}{\partial\hat{b}}$. It is easy to see that $\hat{q}$ satisfies

$$\begin{cases} \hat{q} \in H^1_0(\Omega), \\ \int_\Omega f^3_{\hat{b},\theta}\nabla\hat{q}\cdot\nabla\varphi\,dx + 3\int_\Omega f^2_{\hat{b},\theta}\nabla\hat{p}\cdot\nabla\varphi\,dx = 0 \quad \forall\varphi \in H^1_0(\Omega). \end{cases} \tag{4.23}$$

Taking $\hat{p}$ as test function in the variational formulation of (4.12), (4.13) we get

$$\int_\Omega \hat{p}\,dx = \int_\Omega f_{\hat{b},\theta}^3 |\nabla \hat{p}|^2\,dx + \frac{1}{\theta}\int_\Omega h_0 \frac{\partial \hat{p}}{\partial x_1}\,dx$$

which upon differentiation with respect to $\hat{b}$ yields

$$\int_\Omega \hat{q}\,dx = 3\int_\Omega f_{\hat{b},\theta}^2 |\nabla \hat{p}|^2\,dx + 2\int_\Omega f_{\hat{b},\theta}^3 \nabla \hat{p}\cdot\nabla\hat{q}\,dx + \frac{1}{\theta}\int_\Omega h_0 \frac{\partial \hat{q}}{\partial x_1}\,dx. \tag{4.24}$$

Taking now $\varphi = \hat{p}$ in (4.23) and inserting it into (4.24) gives

$$\int_\Omega \hat{q} = -3\int_\Omega f_{\hat{b},\theta}^2 |\nabla \hat{p}|^2 + \frac{1}{\theta}\int_\Omega h_0 \frac{\partial \hat{q}}{\partial x_1}. \tag{4.25}$$

Since the first term on the right is obviously negative, it remains to show that for $-\theta$ large enough the second term is, in absolute value, bounded by the first one.

By Cauchy–Schwarz inequality we have

$$\left|\int_\Omega h_0(x)\frac{\partial \hat{q}}{\partial x_1}\,dx\right| \leqslant \left[\int_\Omega \frac{h_0^2(x)}{f_{\hat{b},\theta}^3}\,dx\right]^{1/2}\left[\int_\Omega f_{\hat{b},\theta}^3|\nabla\hat{q}|^2\,dx\right]^{1/2}. \tag{4.26}$$

To get a bound for the second factor on the right, we take $\varphi = \hat{q}$ in (4.23) to get

$$\int_\Omega f_{\hat{b},\theta}^3|\nabla\hat{q}|^2\,dx = -3\int_\Omega f_{\hat{b},\theta}^2 \nabla\hat{q}\cdot\nabla\hat{p}\,dx$$

which implies (again by Cauchy–Schwarz)

$$\int_\Omega f_{\hat{b},\theta}^3|\nabla\hat{q}|^2\,dx \leqslant 3\left[\int_\Omega f_{\hat{b},\theta}^3|\nabla\hat{q}|^2\,dx\right]^{1/2}\left[\int_\Omega f_{\hat{b},\theta}|\nabla\hat{p}|^2\,dx\right]^{1/2}$$

and further, since $f_{\hat{b},\theta} \geqslant \hat{b}$

$$\int_\Omega f_{\hat{b},\theta}^3|\nabla\hat{q}|^2\,dx \leqslant \frac{9}{\hat{b}}\int_\Omega f_{\hat{b},\theta}^2|\nabla\hat{p}|^2\,dx. \tag{4.27}$$

We now turn to the first factor in the right-hand side of (4.26). From (4.11) we have

$$f_{\hat{b},\theta}(x) \geqslant \hat{b} + 1 - x_1$$

and thus

$$\int_\Omega \frac{h_0^2(x)}{f_{\hat{b},\theta}^3}\,dx \leqslant 2M_3^2\int_{-1}^{1}\frac{(1-x_1)^{2\alpha}}{(\hat{b}+1-x_1)^3}\,dx_1.$$

Using now $1 - x_1 \leqslant \hat{b} + 1 - x_1$ and $\hat{b} \leqslant 1$, and since $\alpha > 1$, we arrive at

$$\int_\Omega \frac{h_0^2(x)}{f_{\hat{b},\theta}^3}\,dx \leqslant \frac{3^{2\alpha-2}}{\alpha-1}M_3^2. \tag{4.28}$$

Now, inserting (4.27) and (4.28) into (4.26) we have

$$\left|\int_\Omega h_0(x)\frac{\partial \hat q}{\partial x_1}\,dx\right| \leqslant 3^\alpha \frac{M_3}{\sqrt{\hat b}\sqrt{\alpha-1}}\left[\int_\Omega f_{\hat b,\theta}^2|\nabla \hat p|^2\,dx\right]^{1/2}$$

which, together with (4.25) leads to

$$\int_\Omega \hat q\,dx \leqslant \left[\int_\Omega f_{\hat b,\theta}^2|\nabla \hat p|^2\,dx\right]^{1/2}\left\{-3\left[\int_\Omega f_{\hat b,\theta}^2|\nabla \hat p|^2\,dx\right]^{1/2} + \frac{1}{|\theta|\sqrt{\hat b}}3^\alpha\frac{M_3}{\sqrt{\alpha-1}}\right\}.$$

It only remains to insert the estimation of Lemma 4.3 inside the braces on the right-hand side of the previous inequality to end up with

$$\int_\Omega \hat q\,dx \leqslant \left[\int_\Omega f_{\hat b,\theta}^2|\nabla \hat p|^2\,dx\right]^{1/2}\frac{1}{\sqrt{\hat b}}\left\{-3\sqrt{c_0} + \frac{1}{|\theta|}3^\alpha\frac{M_3}{\sqrt{\alpha-1}}\right\}$$

which implies $\int_\Omega \hat q\,dx < 0$ and completes the proof of step 1.

Step 2. We now show that, for $-\theta$ large enough, any solution $(\hat p,\hat b)$ of (4.12)–(4.14) necessarily satisfies $\hat b \leqslant b_0$. For this purpose, we apply Proposition 2.2 with $H_0(x) = 1 - x_1 - \frac{h_0(x)}{\theta}$ and $A = \hat b$, which gives, for any $\theta < -1$ (for example):

$$\int_\Omega \hat p\,dx \leqslant \frac{C_3}{\hat b^3},$$

where $C_3 > 0$ depends just on h_0. Thus, if $\theta < -\sqrt{\frac{C_3}{Fb_0^3}}$ condition (4.14) can only be satisfied if $\hat b \leqslant b_0$.

To complete the proof it only remains to take

$$\theta_1 = \min\left\{-3^{\alpha-1}\frac{M_3}{\sqrt{c_0(\alpha-1)}}, -\sqrt{\frac{C_3}{Fb_0^3}}, -1\right\}. \tag{4.29}$$

This choice guarantees (from step 2), that any $\hat b$ satisfying $\int_\Omega \hat p(\hat b,\theta)\,dx = \theta^2 F$ (of which at least one exists by virtue of Proposition 4.1) satisfies $\hat b \in [0, b_0]$. This same choice of θ_1 also guarantees that $\int_\Omega \hat p(\hat b,\theta)\,dx$ is a monotonous decreasing function of $\hat b$ in $[0, b_0]$ (from step 1). Uniqueness is thus evident. □

Proposition 4.1 shows that there exists, for any given $\theta < 0$, at least one $b > 0$ such that $(p(b,\theta), b)$ is a solution to the system (4.1)–(4.3). Proposition 4.4 allows us to further define such b as a function $b : \theta \in]-\infty, \theta_1[\to b = b(\theta)$. It is thus now possible to introduce, as in the one-dimensional case, the application $S : \theta \in]-\infty, \theta_1[\to S(\theta) = g_2(b(\theta), \theta)$. Notice that the existence of a solution θ of

$$S(\theta) = 0$$

automatically implies that $(b(\theta), \theta)$ is a solution (4.8), (4.9). We are now in a position to state the main result of this section:

Theorem 4.5. *For any $F > 0$ there exists $\bar x_1 \in]-1, 1[$ such that $\forall x_1^0 \in]\bar x_1, 1[$ there exists at least one solution (p, b, θ) of system* (4.1)–(4.6)

Proof. We show, as in Proposition 3.3, that the function S is of class C^∞. Applying Lemma 4.2 we have

$$\int_\Omega (1-x_1)\hat{p}(x)\,dx \leqslant 2$$

which implies, since $\hat{p} = p\theta^2$, that

$$\int_\Omega (1-x_1)p(x)\,dx \to 0, \quad \text{as } \theta \to -\infty.$$

But

$$S(\theta) = \left(1-x_1^0\right)F - \int_\Omega (1-x_1)p(x)\,dx$$

and thus

$$\lim_{\theta\to-\infty} S(\theta) > 0 \quad \forall x_1^0 \in \left]-1,1\right[.$$

Writing, as in Theorem 3.5,

$$S(\theta) = \left(\xi_1^\theta - x_1^0\right)F \quad \text{with } \xi_1^\theta = \frac{\int_\Omega x_1 p\,dx}{\int_\Omega p\,dx} \in \left]-1,1\right[$$

and taking $\bar{x}_1 = \inf_{\theta<\theta_1} \xi_1^\theta \in [-1,1[$ we have, from the definition of ξ_1^θ, that there exists some $\bar{\theta} < \theta_1$ (given by (4.29)) such that $S(\bar{\theta}) < 0$ if $x_1^0 > \bar{x}_1$. The intermediate value theorem thus guarantees that, for some $\theta \in \left]-\infty, \bar{\theta}\right[$, $S(\theta) = 0$. □

Remark 4.6. Theorem 4.5 proves the existence of an equilibrium position for square articulated sliders of general shape, under hypotheses (H_1) and (H_2). It is quite evident that the result generalizes to rectangular sliders. Moreover, it is easy to see that the proof carries out to any slider in which the (nondimensional) domain Ω satisfies

$$\left]\delta_0,1\right[\times \left]-\delta_1,\delta_1\right[\subset \Omega \subset \left]-1,1\right[\times \left]-1,1\right[\tag{4.30}$$

for some positive δ_0 and δ_1. This includes practically all domains which have a downstream edge perpendicular to the sliding direction.

Remark 4.7. It is interesting to notice that Theorems 3.5 and 4.5 establish the existence for each given applied force F of some $\bar{x}_1$ such that, if the slider is articulated at any $x_1^0 > \bar{x}_1$, an equilibrium position exists. Unfortunately, we could not characterize the way $\bar{x}_1$ depends on F. Let us elaborate on this. For θ fixed, we defined

$$\xi_1^\theta = \frac{\int_\Omega x_1 p\,dx}{\int_\Omega p\,dx},$$

where the pressure field p is calculated with the vertical position of the slider such that equilibrium of forces is satisfied (i.e., $\int_\Omega p\,dx = F$). It is reasonable to think, though it remains a conjecture, that ξ_1^θ is a monotonous decreasing function of θ. This is because one expects that as θ decreases towards $-\infty$ the center of pressure ξ_1^θ moves towards the downstream edge $x_1 = 1$. If this conjecture did hold, then we would have

$$\bar{x}_1 = \inf_{\theta<\theta_1} \xi_1^\theta \geqslant \inf_{\theta\leqslant 0} \xi_1^\theta = \xi_1^0.$$

The asymptotic behavior of ξ_1^0 as F tends to $+\infty$ has been analyzed by Ciuperca et al. [7], and in fact for all h_0 not identically equal to zero it can be shown that ξ_1^0 tends to $+1$. The conjectured monotonicity of ξ_1^θ implies

$$\lim_{F\to\infty} \bar{x}_1 = +1.$$

This behavior is qualitatively different from the planar case ($h_0 \equiv 0$), in which x_1 is independent of F (see Section 5). For a nonplanar articulated slider with the articulation located at any position $x_1^0 \in]-1, 1[$ there would thus exist, assuming the conjecture true, a limit force $\overline{F}$ such that for $F > \overline{F}$ no solution of (4.1)–(4.6) exists. This qualitative difference with the planar case should be further explored, since most design methodologies for tilting-pad bearings are based on properties of the (known) solution to the planar case, for which the equilibrium does not "disappear" at high loads.

5. The planar case ($h_0 \equiv 0$)

In the previous section we proved the existence of an equilibrium solution in the 2D case by showing that, for $-\theta$ large enough, the effect of the actual shape h_0 of the bearing is negligible. In this section we consider the case in which h_0, in fact, vanishes, or, stated otherwise, the case in which both surfaces of the bearing are planes. The results hold for both 1D and 2D and are of course stronger and more precise than in the general case.

We thus consider system (1.2)–(1.5) with $h_0 \equiv 0$. Dividing (1.2) by a^3 and denoting $s = -\frac{\theta}{a}$ with $q = a^2 p$ we obtain the following system for the unknowns (q, a, s)

$$\nabla \cdot \big((1 - s x_1)^3 \nabla q\big) = -s \quad \text{in } \Omega, \tag{5.1}$$

$$q = 0 \quad \text{on } \partial\Omega, \tag{5.2}$$

$$\int_\Omega q\,dx = a^2 F, \tag{5.3}$$

$$\int_\Omega q\big(x_1 - x_1^0\big)\,dx = 0 \tag{5.4}$$

with the constraints $a > 0$ and $0 < s < 1$. With this change of variables the problem is thus decoupled: It reduces to finding $q(x)$ and $s \in]0, 1[$ satisfying (5.1), (5.2) and (5.4). Once q and s are found, and since $\int_\Omega q\,dx > 0$, a is obtained from

$$a = \sqrt{\frac{\int_\Omega q\,dx}{F}}.$$

It is thus natural to introduce the function

$$g(s) = \int_\Omega q(s)\big(x_1 - x_1^0\big)\,dx$$

with $q(s) = q$ the unique solution of (5.1), (5.2). The existence problem reduces to finding $s \in]0, 1[$ such that $g(s) = 0$. It is easy to see that g is well defined and of class C^∞ for $s \in]-1, 1[$. It is also evident that $g(0) = 0$. We assume that the domain Ω, not necessarily a rectangle, satisfies (4.30), and we define φ_0 as the (unique) solution of

$$-\nabla^2 \varphi_0 = 1 \tag{5.5}$$

with homogeneous Dirichlet boundary data. Further, we define

$$\bar{x}_1 = \frac{\int_\Omega x_1 \varphi_0 \, dx}{\int_\Omega \varphi_0 \, dx}. \tag{5.6}$$

Notice that $\bar{x}_1$ is the x_1-coordinate of the barycenter of the domain considered with density φ_0. If the domain is symmetric with respect to the reflection $x_1 \to -x_1$, then $\bar{x}_1 = 0$ (as is well known for rectangular bearings).

The main result of this section is:

Theorem 5.1. *If $h_0 \equiv 0$, then for any $x_1^0 \in]\bar{x}_1, 1[$ and any $F > 0$ problem* (1.2)–(1.5) *admits a solution.*

Proof. The proof is straightforward. It consists of two steps: First we show that $g(s) \to +\infty$ as $s \to 1$; then we show that $g'(0) < 0$. Since $g(0) = 0$ this implies the result.

Step 1. Let us thus start by showing that

$$\lim_{s \to 1} g(s) = +\infty. \tag{5.7}$$

For this purpose, let us divide (5.1) by s^3 and define $\hat{s} = \frac{1}{s} - 1$ and $\hat{q} = s^2 q$. System (5.1), (5.2) then becomes

$$\begin{cases} \nabla \cdot ((\hat{s} + 1 - x_1)^3 \nabla \hat{q}) = -1 & \text{in } \Omega, \\ \hat{q} = 0 & \text{on } \partial\Omega. \end{cases}$$

Notice that $\lim_{s \to 1,\ s<1} \hat{s} = 0$. We will prove that

$$\lim_{\hat{s} \to 0} \int_\Omega (x_1 - x_1^0) \hat{q} \, dx = +\infty \tag{5.8}$$

which implies (5.7). Let us choose $\eta_0 > 0$ such that $x_1^0 < 1 - \eta_0$ and write

$$\int_\Omega (x_1 - x_1^0) \hat{q} \, dx = E_1 + E_2 \tag{5.9}$$

with $E_1 = \int_{\Omega_{\eta_0}} (x_1 - x_1^0) \hat{q} \, dx$ and $E_2 = \int_{\Omega - \Omega_{\eta_0}} (x_1 - x_1^0) \hat{q} \, dx$. We first notice that

$$E_1 \geqslant (1 - \eta_0 - x_1^0) \int_{\Omega_{\eta_0}} \hat{q} \, dx.$$

Applying Proposition 2.3 with $H_0(x) = 1 - x_1$ and $A = \hat{s}$ we have

$$\lim_{\hat{s} \to 0} \int_{\Omega_{\eta_0}} \hat{q} \, dx = +\infty$$

which implies

$$\lim_{\hat{s} \to 0} E_1 = +\infty. \tag{5.10}$$

Turning now to E_2, it is immediate from the maximum principle that

$$|\hat{q}(x)| \leqslant \int_{-1}^{x_1} \frac{dy}{(\hat{s} + 1 - y)^2} \quad \forall x \in \Omega$$

implying

$$\left|\hat{q}(x)\right| \leqslant \frac{1}{(\hat{s}+1-x_1)} \quad \forall x \in \Omega$$

and thus

$$0 \leqslant \hat{q}(x) \leqslant \frac{1}{\eta_0} \quad \forall x \in \Omega - \Omega_{\eta_0}.$$

Since $\hat{q} \geqslant 0$,

$$|E_2| \leqslant \frac{4}{\eta_0}. \tag{5.11}$$

Combining (5.9)–(5.11) we get (5.8).

Step 2. We now show that $g'(0) < 0$. By differentiating (5.1) with respect to s it is immediate that the function φ_0 introduced above satisfies

$$\varphi_0 = \frac{dq}{ds}(0)$$

and thus $g'(0) = \int_\Omega \varphi_0\,(x_1 - x_1^0)\,dx$. The claim is obvious from the definition of $\bar{x}_1$, Eq. (5.6). □

Remark 5.2. In all three cases studied above the uniqueness remains an open problem.

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Appendix A. Normalization of variables and normalized distance

Let us denote by the hat symbol ("$\widehat{\ }$") the non-normalized physical quantities. Hydrodynamic contact is assumed to occur in $\widehat{\Omega} =]-L, L[^2$. The upper surface is assumed located at $\hat{x}_3 = \hat{h}_0(\hat{x})$, $\hat{x} \in \widehat{\Omega}$. We denote by h_v a characteristic distance between the bodies and we set $\epsilon = \frac{h_v}{L}$ (assumed very small so that the thin-film hypothesis holds). The upper surface rotates an angle $\hat{\theta}$ (again assumed very small) around the axis $\{\hat{x}_1 = \hat{x}_1^0\}$, and translates a distance $\hat{b}$ (very small too) along the vertical axis $O\hat{x}_3$. In this article we suppose that $\hat{h}_0(\hat{x}_1^0, \hat{x}_2)$ is independent of $\hat{x}_2$. Denoting by $(\hat{x}_1, \hat{x}_2, \hat{x}_3 = h_0(\hat{x}))$ an arbitrary point in the upper surface and by $(\hat{x}_1', \hat{x}_2', \hat{x}_3')$ the new position of this point after rotation and translation, we have

$$\begin{cases} \hat{x}_1' - \hat{x}_1^0 = (\hat{x}_1 - \hat{x}_1^0)\cos\hat{\theta} - (\hat{h}_0(\hat{x}) - \hat{h}_0(\hat{x}^0))\sin\hat{\theta}, \\ \hat{x}_2' = \hat{x}_2, \\ \hat{x}_3' - \hat{h}_0(\hat{x}^0) = (\hat{x}_1 - \hat{x}_1^0)\sin\hat{\theta} + (\hat{h}_0(\hat{x}) - \hat{h}_0(\hat{x}^0))\cos\hat{\theta} + \hat{b}. \end{cases} \tag{A.1}$$

In lubrication theory the following normalization is customary:

$$\theta = \frac{\hat{\theta}}{\epsilon}, \qquad b = \frac{\hat{b}}{\epsilon L}, \qquad x = \frac{\hat{x}}{L}, \qquad x_3 = \frac{\hat{x}_3}{\epsilon L},$$

so that

$$h_0(x) = \frac{\hat{h}_0(Lx)}{\epsilon L}, \qquad x_1^0 = \frac{\hat{x}_1^0}{L}, \qquad \Omega =]-1, 1[^2.$$

From (A.1) and since ϵ is very small we have

$$\begin{cases} L(x_1' - x_1^0) = L(x_1 - x_1^0) + O(\epsilon^2), \\ x_2' = x_2, \\ \epsilon L(x_3' - h_0(x^0)) = \epsilon L(x_1' - x_1^0)\theta + \epsilon L(h_0(x) - h_0(x^0)) + \epsilon Lb + O(\epsilon^3). \end{cases} \tag{A.2}$$

Neglecting the terms of order 2 and using the change of variables $a = b - x_1^0\theta$, we obtain the (rotated and translated) form of the upper surface as:

$$h(x) = h_0(x) + \theta x_1 + a.$$

The domain of definition of h is also $\Omega =]-1, 1[^2$ (unchanged in spite of rotation because we neglect the terms of order ϵ^2).

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